INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 23, 193–220 (1996)

MODELLING OF BYPASS TRANSITION WITH CONDITIONED NAVIER–STOKES EQUATIONS COUPLED TO AN INTERMITTENCY TRANSPORT EQUATION

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SUMMARY

A differential method is proposed to simulate bypass transition. The intermittency in the transition zone is taken into account by conditioned averages. These are averages taken during the fraction of time the flow is turbulent or laminar respectively. Starting from the Navier–Stokes equations, conditioned continuity, momentum and energy equations are derived for the laminar and turbulent parts of the intermittent flow. The turbulence is described by a classical $k-\epsilon$ model. The supplementary parameter, the intermittency factor, is determined by a transport equation applicable for zero, favourable and adverse pressure gradients. Results for these pressure gradients are given.

KEY WORDS transition; turbulence; intermittency; conditioning

INTRODUCTION

In most cases the boundary layer flow around a body is laminar downstream of the stagnation point. Through the presence of perturbations in the flow, however, the laminar boundary layer can change towards a turbulent layer and the length of the transition zone can be considerable. For a turbine blade, its length can extend up to 80% of the chord.[1] Through the transition zone, both the skin friction and the heat transfer may increase several fold. It is clear that one should take into account this transition phenomenon to evaluate quantities such as heat transfer and pressure losses. The laminar phase, which precedes the transition, can be calculated rather easily. The turbulent phase which follows the transition zone, is much more complex to describe. Nevertheless, during recent years turbulence modelling has progressively improved, so that nowadays turbulent flows can be calculated quite accurately. As a result of its complexity, this is not yet the case for the transition zone. The main reason is that besides the simultaneous presence of turbulent and laminar flow, there is also the interaction between the two phases.

Different modes of transition exist. The first mode is *natural transition* and is typical of external flows. Starting from a certain Reynolds number, based on the displacement thickness $\delta_1, Re_{\delta_1} = U_\infty \delta_1/\nu$, the laminar layer is susceptible to small perturbations. By linearizing the boundary layer equations, Orr and Sommerfeld[2] showed that a sinusoidal velocity perturbation is amplified throughout a range of wave numbers. These unstable waves are named after Tollmien and Schlichting (TS) and can be represented by two-dimensional vortex tubes. The evolution of these waves finally leads to the creation of turbulent Emmons spots. In the presence of large perturbations,

CCC 0271-2091/96/030193-28

Received June 1995
Revised November 1995

such as freestream turbulence, the creation of TS waves is bypassed. The perturbations induce the turbulent spots directly, through diffusion of turbulent eddies. This *bypass transition* is typical of gas turbines, where freestream turbulence levels can reach up to 20% or more. During transition, the flow changes gradually from laminar to turbulent. The two phases exist together and alternate as a function of time. The relative fraction of time the flow is turbulent at a certain position is called the intermittency factor γ. This factor evolves from 0% to 100%. From experiments in zero, favourable and adverse pressure gradient flows[3,4], the evolution of γ reveals a universal character.

In spite of the universal character of the growth of turbulence, many transition models have been developed without making use of the intermittency factor γ. The classical integral method of Goldstein[5] even neglects the length of transition. An abrupt switch from laminar to turbulent flow is imposed at the transition point. The connection between the two states is guaranteed through the equality of momentum thickness, $\theta_l(x_{tr}) = \theta_t(x_{tr})$. This oversimplified method predicts large jumps in skin friction and heat transfer which cannot be tolerated for designing turbine blades. In the one-equation approach of Rodi[6] and Fujisawa *et al.*[7] a gradual change from a laminar to a turbulent layer is obtained by modifying the mixing length. By this gradual change in the streamwise direction, the typical peak values of $\overline{u_t'^2}$ and $\overline{u_t'v_t'}$ during transition cannot be reached. Singer[8] explains that the proposed correlations cannot be generally applicable. When using the low-Reynolds-number version of the two-equation turbulence models, a transition is produced which occurs in most cases too early and too fast in comparison with the physical transition.[9] Schmidt and Patankar[10] tried to modify the production term for the k–ϵ model in order to match the predicted transition with the physical transition. Wilcox[11] tried to reach the same goal for a k–ω model but by modifying the damping functions. For both approaches the physical peak values of $\overline{u_t'^2}$ and $\overline{u_t'v_t'}$ cannot be reproduced. Moreover, the correlations used are very sensitive and are in principle not very practical for general use.

All the models mentioned above lack generality, as the presence of the turbulent spots in the laminar phase has not been taken into account. The model of Dhawan and Narasimha[12] and Dey and Narasimha[13] treats the global flow as a linear combination of a laminar and turbulent flow according to the ratio $(1-\gamma)/\gamma$. This method guarantees a good prediction of skin friction and heat transfer for favourable and zero pressure gradient flow. Owing to the neglect of the existing interaction of the turbulent spots with the surrounding laminar flow, the method cannot predict transitional flow with adverse pressure gradient. The laminar state separates as soon as a positive pressure gradient is applied. In reality the entrainment of the laminar phase by the more stable turbulent flow prevents this instantaneous separation. For the same reason, the $\overline{u_t'^2}$ and $\overline{u_t'v_t'}$ profiles are underestimated.

An improvement of the existing transition models necessitates the incorporation of the interactive forces between laminar and turbulent phases. To make the distinction between these phases during the transition, conditioned averages can be used. These are averages taken during the fraction of time the flow is turbulent or laminar respectively.

The technique of conditioning the flow equations in intermittent flows was introduced by Libby[14] and further refined by Dopazo.[15] This early work concentrated on the intermittency in the outer edge region of turbulent shear flows. Subsequently, many researchers have developed models for the terms generated by the interaction between the turbulent and non-turbulent parts of the flow, as well as a transport equation for the intermittency factor, for use in the free boundary of shear layers. A recent example is the work by Cho and Chung.[16] More examples are referenced in this work.

Conditionally averaged equations can also be used to describe the intermittency during the transition of a boundary layer flow. This technique was introduced by Vancoillie and Dick.[17] In their work, the highly mathematical technique of Libby[14] was followed very closely. The interaction terms were modelled in an *ad hoc* way appropriate for boundary layer flows on a flat plate. Good results were obtained for zero, favourable and mild adverse pressure gradients. The work focused on low-turbulence-level natural transition. We focus here on bypass transition since our work is motivated by turbomachinery applications. In this paper we derive the conditioned equations in a more physical way than in the work of Libby. The result is a set of equations in which the laminar–turbulent interaction terms need no modelling. Solely the prescription of the intermittency factor is needed.

CONDITIONED AVERAGES

To illustrate the typical behaviour of a flow quantity in transitional flow, Figure 1 shows a sketch of the evolution in time of the velocity component in the x-direction, u. We can make the distinction between phases where the flow is turbulent, characterized by high-frequency fluctuations of the flow quantity, and phases where the flow is non-turbulent, characterized by low-frequency fluctuations.

The succession in time of turbulent and non-turbulent behaviour of the flow in a transitional region is called intermittency. The intermittency factor γ at some point in the flow is the relative fraction of the time the flow is turbulent. In an unsteady flow this factor may change with time. At a given time t, spatially some regions in the flow are turbulent, other regions are non-turbulent. Between these regions there are interfaces where flow quantities may change very abruptly.

In order to take into account the intermittent behaviour of the flow, we define an intermittency function $I(x, y, z, t)$ with value 1 in a turbulent region and value 0 in a non-turbulent region (Figure 1). The time-averaged value of this function during some time interval T is the intermittency factor

$$\gamma = \frac{1}{T}\int_0^T I(x, y, z, t)\,\mathrm{d}t = \gamma(x, y, z, t).$$

The time interval T is chosen to be large with respect to the time scales of the turbulence, but still small with respect to the time scales of the mean flow.

In the following we will describe the flow by considering averages and fluctuations during the turbulent and non-turbulent fractions of the time. We call these quantities conditioned averages and fluctuations. Furthermore, we will denote non-turbulent regions in the flow as laminar although this designation is not universally accepted, since even in a globally steady flow, fluctuations are present in the non-turbulent regions.

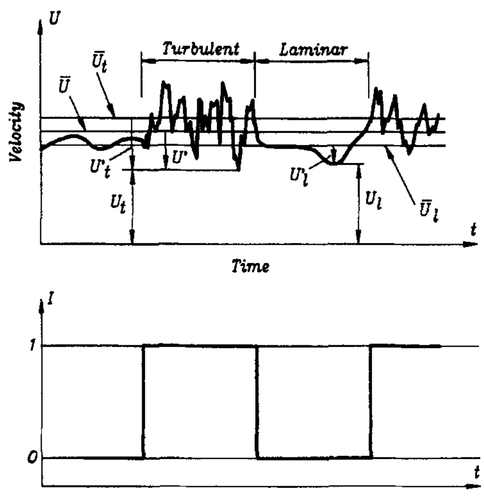

Figure 1. Typical hot-wire signal for velocity, with corresponding intermittency function I

We calculate first the turbulent and laminar conditioned mean values of a quantity for Reynolds averaging. Afterwards we verify the results for Favre averaging. As an example we take the velocity component in the x-direction. This quantity can be decomposed into mean and fluctuating components by

$$
\begin{aligned}
u &= \overline{u} + u' && \text{globally},\\
u &= \overline{u}_t + u'_t && \text{for } I = 1,\\
u &= \overline{u}_l + u'_l && \text{for } I = 0.
\end{aligned}
$$

The conditioned turbulent mean value and fluctuation satisfy respectively

$$\overline{Iu} = \gamma\overline{u}_t = \frac{1}{T}\int_0^T Iu\,\mathrm{d}t \quad \text{and} \quad \overline{Iu'_t} = 0.$$

The conditioned laminar mean value and fluctuation satisfy respectively

$$\overline{(1-I)u} = (1-\gamma)\overline{u}_l = \frac{1}{T}\int_0^T (1-I)u\,\mathrm{d}t \quad \text{and} \quad \overline{(1-I)u'_l} = 0.$$

The global mean value satisfies

$$\overline{u} = \frac{1}{T}\int_0^T [(1-I)u_l + Iu_t]\mathrm{d}t = (1-\gamma)\overline{u}_l + \gamma\overline{u}_t.$$

As expected, the global mean value is a linear combination of the laminar and turbulent mean values. A global mean value of a product, on the other hand, cannot be written as a linear combination of the mean laminar and turbulent products. For the global Reynolds stress during the turbulent phase, $-\overline{(u'v'_t)}_t$, the following steps can be derived during the turbulent phase ($I = 1$):

$$
\begin{aligned}
u' &= u - \overline{u}\\
&= \overline{u}_t - \overline{u} + u'_t\\
&= \overline{u}_t - \gamma\overline{u}_t - (1-\gamma)\overline{u}_l + u'_t\\
&= (1-\gamma)(\overline{u}_t - \overline{u}_l) + u'_t,\\
v' &= (1-\gamma)(\overline{v}_t - \overline{v}_l) + v'_t.
\end{aligned}
$$

Multiplication of u' and v' and time averaging leads to

$$\overline{(u'v')}_t = \overline{u'_t v'_t} + (1-\gamma)^2(\overline{u}_t - \overline{u}_l)(\overline{v}_t - \overline{v}_l).$$

A similar derivation during the laminar phase ($I = 0$) gives

$$
\begin{aligned}
u' &= u - \overline{u}\\
&= \overline{u}_l - \overline{u} + u'_l\\
&= \overline{u}_l - \gamma\overline{u}_t - (1-\gamma)\overline{u}_l + u'_l\\
&= \gamma(\overline{u}_l - \overline{u}_t) + u'_l,\\
v' &= \gamma(\overline{v}_l - \overline{v}_t) + v'_l.
\end{aligned}
$$

Multiplication of u' and v' and time averaging leads to

$$\overline{(u'v')}_l = \overline{u'_l v'_l} + \gamma^2(\overline{u}_l - \overline{u}_t)(\overline{v}_l - \overline{v}_t).$$

The global mean Reynolds stress $-\overline{u'v'}$ can be written as a function of the laminar and turbulent parts by

$$\begin{aligned}\overline{u'v'} &= (1-\gamma)\overline{(u'v')}_l + \gamma\overline{(u'v')}_t \\ &= (1-\gamma)\overline{u_l'v_l'} + \gamma\overline{u_t'v_t'} + \gamma(1-\gamma)(\overline{u}_l - \overline{u}_t)(\overline{v}_l - \overline{v}_t).\end{aligned} \tag{1}$$

For $\overline{u'^2}$ one obtains

$$\overline{u'^2} = (1-\gamma)\overline{u_l'^2} + \gamma\overline{u_t'^2} + \gamma(1-\gamma)(\overline{u}_l - \overline{u}_t)^2. \tag{2}$$

Besides the linear combination terms, a third term arises which is premultiplied by the coefficient $\gamma(1-\gamma)$. This coefficient is zero before and after transition and reaches a maximum for $\gamma = 0{\cdot}5$. In equation (2), one notices that this extra term is always positive and leads to a rise in turbulent kinetic energy k. The term originates from the alternating velocity profiles near the interfaces of the turbulent and laminar regions and can be seen as a large turbulent eddy. In Figure 2 typical laminar and turbulent velocity profiles are drawn at a certain position in the transition zone. The extra term in (2) leads, in the case of the turbulent kinetic energy, to higher levels close to and further away from the wall.

Further, we derive the equations for the conditioned averages of a space or time derivative quantity. The turbulent conditioned average of a space derivative term $\partial u/\partial x$ is defined by $\overline{I\partial u/\partial x}$. During the turbulent phase we decompose by

$$u = \overline{u}_t + u_t' \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{\partial \overline{u}_t}{\partial x} + \frac{\partial u_t'}{\partial x}.$$

We accept that the quantity $\partial u_t'/\partial x$ is uncorrelated, like u_t', such that the time average during the turbulent phase is zero:

$$\overline{I\frac{\partial u_t'}{\partial x}} = 0.$$

As a consequence, the contribution of the turbulent phase in the turbulent mean value is $\gamma\partial\overline{u}_t/\partial x$. Furthermore, there is a contribution coming from the fronts between turbulent and laminar regions. Figure 3 shows schematically the passage of an upgoing front, i.e. a front where the state changes from laminar to turbulent. Quantities at the front position are supposed to vary linearly over a very short distance δx_1.

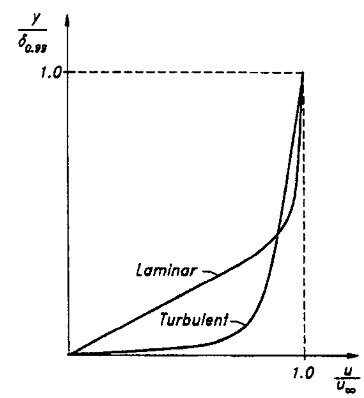

Figure 2. Laminar and turbulent velocity profiles at a certain position in the transition zone

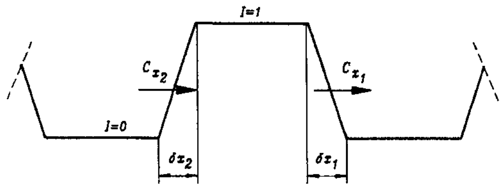

Figure 3. Representation of a turbulent region by its intermittency function

The intermittency function I can be written as

$$I = \frac{t}{\delta t_1} \quad \text{for } 0 < t < \delta t_1,$$

where δt_1 denotes the time the front needs to pass. The velocity can be written as

$$u = (\bar{u}_t + u_t')\left(1 - \frac{x}{\delta x_1}\right) + (\bar{u}_l + u_l')\frac{x}{\delta x_1} \quad \text{for} \quad 0 < x < \delta x_1.$$

The space derivative is

$$\frac{\partial u}{\partial x} = \frac{\bar{u}_l + u_l' - \bar{u}_t - u_t'}{\delta x_1}.$$

The velocity of the upgoing front, c_{x_1}, is

$$c_{x_1} = \frac{\delta x_1}{\delta t_1} \quad \text{for } \delta t_1 \to 0, \delta x_1 \to 0.$$

For one passage of an upgoing front during the averaging time T, the contribution to the conditioned average of a derivative is

$$\frac{1}{T}\int_0^{\delta t_1} I \frac{\bar{u}_l - \bar{u}_t}{\delta x_1}\,\mathrm{d}t + \frac{1}{T}\int_0^{\delta t_1} I \frac{u_l' - u_t'}{\delta x_1}\,\mathrm{d}t.$$

The contribution from the fluctuating terms is of higher order and tends to zero for $\delta t_1 \to 0$ and $\delta x_1 \to 0$. Thus the resulting contribution is

$$\frac{1}{T}\int_0^{\delta t_1} \frac{t}{\delta t_1}\frac{\bar{u}_l - \bar{u}_t}{\delta x_1}\,\mathrm{d}t = \frac{1}{T}\frac{1}{\delta t_1}\frac{\bar{u}_l - \bar{u}_t}{\delta x_1}\frac{(\delta t_1)^2}{2} = \frac{1}{T}\frac{\bar{u}_l - \bar{u}_t}{2}\frac{\delta t_1}{\delta x_1} = \frac{1}{T}\frac{\bar{u}_l - \bar{u}_t}{2}\frac{1}{c_{x_1}}. \tag{3}$$

For the downgoing front the contribution is similar. The contribution of both fronts for the passage of a single turbulent region is then

$$\frac{1}{2}\frac{1}{T}(\bar{u}_t - \bar{u}_l)\left(\frac{1}{c_{x_2}} - \frac{1}{c_{x_1}}\right), \tag{4}$$

where c_{x_2} is the velocity of the downgoing front. By integrating over many passages, a sum of terms of form (4) appears. The interpretation of this sum is straightforward. We consider the definition of the intermittency factor on the position P(x, y, z) and on a position P$'$ an infinitesimal distance δx further in the x-direction. When the upgoing front passes at time t_1 at the position P, it passes at time $t_1 + \delta x / c_{x_1}$

at the position P′. Similarly, the downgoing front passes at times t_2 and $t_2 + \delta x/c_{x_2}$. Neglecting higher-order variations of c_{x_1} and c_{x_2} during the passage, the fraction of time that turbulent flow is seen at point P is given by

$$\gamma = \frac{\sum (t_2 - t_1)}{T},$$

while at point P′ it is

$$\gamma + \delta\gamma = \frac{\sum (t_2 - t_1)}{T} + \frac{\delta x}{T}\sum\left(\frac{1}{c_{x_2}} - \frac{1}{c_{x_1}}\right).$$

Hence

$$\frac{\partial\gamma}{\partial x} = \frac{1}{T}\sum\left(\frac{1}{c_{x_2}} - \frac{1}{c_{x_1}}\right).$$

This results in the rule for a space derivative

$$\overline{I\frac{\partial u}{\partial x}} = \gamma\frac{\partial \overline{u}_t}{\partial x} + \tfrac{1}{2}(\overline{u}_t - \overline{u}_l)\frac{\partial\gamma}{\partial x}. \tag{5}$$

This rule is valid for every other space direction.

The laminar conditioned mean value can be derived similarly as

$$\overline{(1 - I)\frac{\partial u}{\partial x}} = (1 - \gamma)\frac{\partial \overline{u}_l}{\partial x} + \tfrac{1}{2}(\overline{u}_t - \overline{u}_l)\frac{\partial\gamma}{\partial x}. \tag{6}$$

The sum of expressions (5) and (6) gives

$$\overline{\frac{\partial u}{\partial x}} = \frac{\partial[\gamma\overline{u}_t + (1 - \gamma)\overline{u}_l]}{\partial x} = \frac{\partial\overline{u}}{\partial x},$$

which, of course, should be the result.

Following a similar reasoning, conditioned mean values for a time derivative quantity can be constructed. We consider $\overline{I\partial u/\partial t}$. The contribution of the turbulent phase to the integral defining the mean value is $\gamma\partial\overline{u}_t/\partial t$. The front contributions for one passage are respectively

$$\frac{1}{2}\frac{1}{T}(\overline{u}_t - \overline{u}_l) \quad \text{and} \quad \frac{1}{2}\frac{1}{T}(\overline{u}_l - \overline{u}_t).$$

For γ constant in time there is complete compensation of these two terms. For varying γ there is a resultant contribution. Over a time interval T the passage of the upgoing fronts is advanced in the mean by the amount $\frac{1}{2}(\partial\gamma/\partial t)T$, while the passage of the downgoing fronts is retarded by the same amount. Thus over a given time T, for γ augmenting in time, more upgoing fronts pass than downgoing fronts. The resultant contribution of the fronts to the integral is

$$\tfrac{1}{2}(\overline{u}_t - \overline{u}_l)\frac{\partial\gamma}{\partial t}.$$

Thus the time derivative rule is like the space derivative rule:

$$\overline{I\frac{\partial u}{\partial t}} = \gamma\frac{\partial \overline{u}_t}{\partial t} + \tfrac{1}{2}(\overline{u}_t - \overline{u}_l)\frac{\partial\gamma}{\partial t}. \tag{7}$$

Further, we need the conditioned average of the product of a quantity and the space derivative of another quantity, $\overline{Ia\partial b/\partial x}$. The contribution of the turbulent phase to the conditioned average is

$$\gamma\left(\bar{a}_t\frac{\partial \bar{b}_t}{\partial x}+\overline{a'_t\frac{\partial b'_t}{\partial x}}\right).$$

At the upgoing front, linear variation of the quantities is assumed and results in

$$a=(\bar{a}_t+a'_t)\frac{t}{\delta t_1}+(\bar{a}_l+a'_l)\left(1-\frac{t}{\delta t_1}\right),$$
$$\frac{\partial b}{\partial x}=\frac{\bar{b}_l+b'_l-\bar{b}_t-b'_t}{\delta x_1}.$$

The contribution to the integral is

$$[(\tfrac{1}{3}\bar{a}_t+\tfrac{1}{6}\bar{a}_l)(\bar{b}_l-\bar{b}_t)+\tfrac{1}{6}\overline{a'_lb'_l}-\tfrac{1}{3}\overline{a'_tb'_t}]\frac{\delta t_1}{\delta x_1}.$$

The similar contribution from the downgoing front is

$$[(\tfrac{1}{3}\bar{a}_t+\tfrac{1}{6}\bar{a}_l)(\bar{b}_t-\bar{b}_l)+\tfrac{1}{3}\overline{a'_tb'_t}-\tfrac{1}{6}\overline{a'_lb'_l}]\frac{\delta t_2}{\delta x_2}.$$

The final result is

$$\overline{Ia\frac{\partial b}{\partial x}}=\gamma\left(\bar{a}_t\frac{\partial \bar{b}_t}{\partial x}+\overline{a'_t\frac{\partial b'_t}{\partial x}}\right)+(\tfrac{1}{3}\bar{a}_t+\tfrac{1}{6}\bar{a}_l)(\bar{b}_t-\bar{b}_l)\frac{\partial\gamma}{\partial x}+(\tfrac{1}{3}\,\overline{a'_tb'_t}-\tfrac{1}{6}\overline{a'_lb'_l})\frac{\partial\gamma}{\partial x}. \tag{8}$$

CONDITIONED NAVIER–STOKES EQUATIONS

The rules for conditioned mean values and derivatives also carry over to Favre averages. We define mean and fluctuating parts of the density by

$$\rho=\bar{\rho}_t+\rho'_t \quad \text{for } I=1, \quad \text{where } \overline{I\rho}=\gamma\bar{\rho}_t,$$
$$\rho=\bar{\rho}_l+\rho'_l \quad \text{for } I=0, \quad \text{where } \overline{(1-I)\rho}=(1-\gamma)\bar{\rho}_l.$$

Hence $\bar{\rho}=\gamma\bar{\rho}_t+(1-\gamma)\bar{\rho}_l$. Further, the turbulent and laminar Favre averages of the u-velocity are defined by

$$\overline{I\rho u}=\gamma\bar{\rho}_t\tilde{u}_t, \qquad \overline{(1-I)\rho u}=(1-\gamma)\bar{\rho}_l\tilde{u}_l.$$

The global Favre average follows from

$$\overline{\rho u}=\bar{\rho}\tilde{u}=\gamma\bar{\rho}_t\tilde{u}_t+(1-\gamma)\bar{\rho}_l\tilde{u}_l.$$

In the following we neglect the fluctuations during the laminar phase. We introduce this simplification mainly to avoid the need for a model for the laminar fluctuations.

We derive now the conditioned turbulent mean continuity equation. The unaveraged equation is

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}=0.$$

According to the rules for derivatives, we obtain as turbulent conditioned mean equation

$$\gamma \frac{\partial \overline{\rho}_t}{\partial t} + \gamma \frac{\partial \overline{\rho}_t \tilde{u}_t}{\partial x} + \gamma \frac{\partial \overline{\rho}_t \tilde{v}_t}{\partial y} = \tfrac{1}{2} S_\rho^\gamma,$$

with

$$S_\rho^\gamma = (\overline{\rho}_l - \overline{\rho}_t)\frac{\partial \gamma}{\partial t} + (\overline{\rho}_l \tilde{u}_l - \overline{\rho}_t \tilde{u}_t)\frac{\partial \gamma}{\partial x} + (\overline{\rho}_l \tilde{v}_l - \overline{\rho}_t \tilde{v}_t)\frac{\partial \gamma}{\partial y}.$$

The conditioned turbulent continuity equation used in the calculation is

$$\frac{\partial \overline{\rho}_t}{\partial t} + \frac{\partial \overline{\rho}_t \tilde{u}_t}{\partial x} + \frac{\partial \overline{\rho}_t \tilde{v}_t}{\partial y} = \frac{1}{2\gamma} S_\rho^\gamma. \tag{9}$$

The conditioned laminar equation is similarly

$$\frac{\partial \overline{\rho}_l}{\partial t} + \frac{\partial \overline{\rho}_l \tilde{u}_l}{\partial x} + \frac{\partial \overline{\rho}_l \tilde{v}_l}{\partial y} = \frac{1}{2(1-\gamma)} S_\rho^\gamma. \tag{10}$$

By summing (9) and (10) multiplied by γ and $1-\gamma$ respectively, the global continuity equation can be reproduced. To illustrate this, we consider first the time derivative terms:

$$\gamma \frac{\partial \overline{\rho}_t}{\partial t} - \tfrac{1}{2}(\overline{\rho}_l - \overline{\rho}_t)\frac{\partial \gamma}{\partial t} + (1-\gamma)\frac{\partial \overline{\rho}_l}{\partial t} - \tfrac{1}{2}(\overline{\rho}_l - \overline{\rho}_t)\frac{\partial \gamma}{\partial t}.$$

This expression can be reorganized as

$$\gamma \frac{\partial \overline{\rho}_t}{\partial t} + \overline{\rho}_t \frac{\partial \gamma}{\partial t} + (1-\gamma)\frac{\partial \overline{\rho}_l}{\partial t} - \overline{\rho}_l \frac{\partial \gamma}{\partial t} = \frac{\partial \gamma \overline{\rho}_t}{\partial t} + \frac{\partial (1-\gamma)\overline{\rho}_l}{\partial t} = \frac{\partial \overline{\rho}}{\partial t}.$$

A similar combination can be made for the convective terms which finally leads to

$$\frac{\partial[\gamma \overline{\rho}_t + (1-\gamma)\overline{\rho}_l]}{\partial t} + \frac{\partial[\gamma \overline{\rho}_t \tilde{u}_t + (1-\gamma)\overline{\rho}_l \tilde{u}_l]}{\partial x} + \frac{\partial[\gamma \overline{\rho}_t \tilde{v}_t + (1-\gamma)\rho_l \tilde{v}_l]}{\partial y} = 0.$$

This equation represents the global mean continuity equation.

The momentum equations may be treated similarly. We write the momentum equations in compact form as

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial x_j},$$

where the summation convention is used. The terms τ_{ij} denote the molecular stress components. During the turbulent phase the Favre and Reynolds decompositions are

$$\begin{aligned} \rho u_i &= \rho(\tilde{u}_{ti} + u''_{ti}), & \rho u_i u_j &= \rho(\tilde{u}_{ti} + u''_{ti})(\tilde{u}_{tj} + u''_{tj}) \\ p &= \overline{p}_t + p'_t, & \tau_{ij} &= \overline{\tau_{tij}} + \tau'_{tij}. \end{aligned} \tag{11}$$

The turbulent conditioned equations are

$$\gamma\frac{\partial\overline{\rho}_t\tilde{u}_{ti}}{\partial t}+\tfrac{1}{2}(\overline{\rho}_t\tilde{u}_{ti}-\overline{\rho}_l\tilde{u}_{li})\frac{\partial\gamma}{\partial t}+\gamma\frac{\partial\overline{\rho}_t\tilde{u}_{ti}\tilde{u}_{tj}}{\partial x_j}+\gamma\frac{\partial\overline{\rho u''_{ti}u''_{tj}}}{\partial x_j}$$
$$+\tfrac{1}{2}(\overline{\rho}_t\tilde{u}_{ti}\tilde{u}_{tj}+\overline{\rho u''_{ti}u''_{tj}}-\overline{\rho}_l\tilde{u}_{li}\tilde{u}_{lj})\frac{\partial\gamma}{\partial x_j}+\gamma\frac{\partial\overline{p}_t}{\partial x_i}+\tfrac{1}{2}(\overline{p}_t-\overline{p}_l)\frac{\partial\gamma}{\partial x_i}$$
$$=\gamma\frac{\partial\overline{\tau_{tij}}}{\partial x_j}+\tfrac{1}{2}(\overline{\tau_{tij}}-\overline{\tau_{lij}})\frac{\partial\gamma}{\partial x_j}.$$

The usual eddy viscosity modelling approximations are now introduced:

$$\overline{\tau}^{R}_{ij}=-\overline{\rho u''_{ti}u''_{tj}}=-\tfrac{2}{3}\overline{\rho}_t\tilde{k}_t\delta_{ij}+2\mu_e\tilde{S}_{tij},$$
$$\overline{\tau_{tij}}=2\overline{\mu}_t\tilde{S}_{tij},\qquad \rho_t\tilde{k}_t=\tfrac{1}{2}\overline{\rho u''^2_{ti}},$$
$$\tilde{S}_{tij}=\tfrac{1}{2}\left(\frac{\partial\tilde{u}_{ti}}{\partial x_j}+\frac{\partial\tilde{u}_{tj}}{\partial x_i}\right)-\tfrac{1}{3}\delta_{ij}\frac{\partial\tilde{u}_{tk}}{\partial x_k},$$

where $\overline{\tau}^{R}_{ij}$ are the Reynolds stress components, $\tilde{k}_t$ is the turbulence kinetic energy during the turbulent phase, μ_e is the eddy viscosity and $\tilde{S}_{tij}$ is the rate-of-shear tensor based on Favre-averages during the turbulent phase. We neglect here front contributions to the turbulent mean shear stress.

The resulting x-momentum equation is

$$\frac{\partial(\overline{\rho}_t\tilde{u}_t)}{\partial t}+\frac{\partial(\overline{\rho}_t\tilde{u}_t\tilde{u}_t)}{\partial x}+\frac{\partial(\overline{\rho}_t\tilde{u}_t\tilde{v}_t)}{\partial y}+\frac{\partial(\overline{p}_t+\frac{2}{3}\overline{\rho}_t\tilde{k}_t)}{\partial x}=\frac{\partial(\overline{\mu}_t+\mu_e)\tilde{S}_{txx}}{\partial x}+\frac{\partial(\overline{\mu}_t+\mu_e)\tilde{S}_{txy}}{\partial y}+\frac{1}{2\gamma}S^{\gamma}_{\rho u},\tag{12}$$

with

$$S^{\gamma}_{\rho u}=(\overline{\rho}_l\tilde{u}_l-\overline{\rho}_t\tilde{u}_t)\frac{\partial\gamma}{\partial t}+(\overline{\rho}_l\tilde{u}_l\tilde{u}_l-\overline{\rho}_t\tilde{u}_t\tilde{u}_t)\frac{\partial\gamma}{\partial x}+(\overline{\rho}_l\tilde{u}_l\tilde{v}_l-\overline{\rho}_t\tilde{u}_t\tilde{v}_t)\frac{\partial\gamma}{\partial y}$$
$$+\ (\overline{p}_l-\overline{p}_t-\tfrac{2}{3}\overline{\rho}_t\tilde{k}_t)\frac{\partial\gamma}{\partial x}-[\overline{\mu}_l\tilde{S}_{lxx}-(\overline{\mu}_t+\mu_e)\tilde{S}_{txx}]\frac{\partial\gamma}{\partial x}-[\overline{\mu}_l\tilde{S}_{lxy}-(\overline{\mu}_t+\mu_e)\tilde{S}_{txy}]\frac{\partial\gamma}{\partial y}.$$

The corresponding laminar conditioned equation is

$$\frac{\partial(\overline{\rho}_l\tilde{u}_l)}{\partial t}+\frac{\partial(\overline{\rho}_l\tilde{u}_l\tilde{u}_l)}{\partial x}+\frac{\partial(\overline{\rho}_l\tilde{u}_l\tilde{v}_l)}{\partial y}+\frac{\partial\overline{p}_l}{\partial x}=\frac{\partial\overline{\mu}_l\tilde{S}_{lxx}}{\partial x}+\frac{\partial\overline{\mu}_l\tilde{S}_{lxy}}{\partial y}+\frac{1}{2(1-\gamma)}S^{\gamma}_{\rho u}.\tag{13}$$

The y-momentum equations for the turbulent and laminar phases can be derived similarly as

$$\frac{\partial(\overline{\rho}_t\tilde{v}_t)}{\partial t}+\frac{\partial(\overline{\rho}_t\tilde{v}_t\tilde{u}_t)}{\partial x}+\frac{\partial(\overline{\rho}_t\tilde{v}_t\tilde{v}_t)}{\partial y}+\frac{\partial(\overline{p}_t+\frac{2}{3}\overline{\rho}_t\tilde{k}_t)}{\partial y}=\frac{\partial(\overline{\mu}_t+\mu_e)\tilde{S}_{tyx}}{\partial x}+\frac{\partial(\overline{\mu}_t+\mu_e)\tilde{S}_{tyy}}{\partial y}+\frac{1}{2\gamma}S^{\gamma}_{\rho v},\tag{14}$$

$$\frac{\partial(\overline{\rho}_l\tilde{v}_l)}{\partial t}+\frac{\partial(\overline{\rho}_l\tilde{v}_l\tilde{u}_l)}{\partial x}+\frac{\partial(\overline{\rho}_l\tilde{v}_l\tilde{v}_l)}{\partial y}+\frac{\partial\overline{p}_l}{\partial y}=\frac{\partial\overline{\mu}_l\tilde{S}_{lyx}}{\partial x}+\frac{\partial\overline{\mu}_l\tilde{S}_{lyy}}{\partial y}+\frac{1}{2(1-\gamma)}S^{\gamma}_{\rho v},\tag{15}$$

with

$$S^{\gamma}_{\rho v}=(\overline{\rho}_l\tilde{v}_l-\overline{\rho}_t\tilde{v}_t)\frac{\partial\gamma}{\partial t}+(\overline{\rho}_l\tilde{v}_l\tilde{u}_l-\overline{\rho}_t\tilde{v}_t\tilde{u}_t)\frac{\partial\gamma}{\partial x}+(\overline{\rho}_l\tilde{v}_l\tilde{v}_l-\overline{\rho}_t\tilde{v}_t\tilde{v}_t)\frac{\partial\gamma}{\partial y}$$
$$+\ (\overline{p}_l-\overline{p}_t-\tfrac{2}{3}\overline{\rho}_t\tilde{k}_t)\frac{\partial\gamma}{\partial y}-[\overline{\mu}_l\tilde{S}_{lyx}-(\overline{\mu}_t+\mu_e)\tilde{S}_{tyx}]\frac{\partial\gamma}{\partial x}-[\overline{\mu}_l\tilde{S}_{lyy}-(\overline{\mu}_t+\mu_e)\tilde{S}_{tyy}]\frac{\partial\gamma}{\partial y}.$$

The energy equation can be treated in the same way. The result is as for the other equations a laminar and a turbulent equation which are similar to the global averaged equation supplemented with source terms due to the front passages. The resulting energy equations are

$$\frac{\partial(\bar{\rho}_t\tilde{E}_t)}{\partial t}+\frac{\partial(\bar{\rho}_t\tilde{H}_t\tilde{u}_t)}{\partial x}+\frac{\partial(\bar{\rho}_t\tilde{H}_t\tilde{v}_t)}{\partial y}=\frac{\partial(\bar{\tau}^{\rm tot}_{txx}\tilde{u}_t+\bar{\tau}^{\rm tot}_{txy}\tilde{v}_t-\bar{q}^{\rm tot}_{tx})}{\partial x}+\frac{\partial(\bar{\tau}^{\rm tot}_{txy}\tilde{u}_t+\bar{\tau}^{\rm tot}_{tyy}\tilde{v}_t-\bar{q}^{\rm tot}_{ty})}{\partial y}+\frac{1}{2\gamma}S^{\gamma}_{\rho E},$$

$$\frac{\partial(\bar{\rho}_l\tilde{E}_l)}{\partial t}+\frac{\partial(\bar{\rho}_l\tilde{H}_l\tilde{u}_l)}{\partial x}+\frac{\partial(\bar{\rho}_l\tilde{H}_l\tilde{v}_l)}{\partial y}=\frac{\partial(\bar{\tau}_{lxx}\tilde{u}_l+\bar{\tau}_{lxy}\tilde{v}_l-\bar{q}_{lx})}{\partial x}+\frac{\partial(\bar{\tau}_{lxy}\tilde{u}_l+\bar{\tau}_{lyy}\tilde{v}_l-\bar{q}_{ly})}{\partial y}+\frac{1}{2(1-\gamma)}S^{\gamma}_{\rho E},$$

with

$$\begin{aligned}S^{\gamma}_{\rho E}=&(\bar{\rho}_l\tilde{E}_l-\bar{\rho}_t\tilde{E}_t)\frac{\partial\gamma}{\partial t}+(\bar{\rho}_l\tilde{H}_l\tilde{u}_l-\bar{\rho}_t\tilde{H}_t\tilde{u}_t)\frac{\partial\gamma}{\partial x}+(\bar{\rho}_l\tilde{H}_l\tilde{v}_l-\bar{\rho}_t\tilde{H}_t\tilde{v}_t)\frac{\partial\gamma}{\partial y}\\&-(\bar{\tau}_{lxx}\tilde{u}_l+\bar{\tau}_{lxy}\tilde{v}_l-\bar{\tau}^{\rm tot}_{txx}\tilde{u}_t-\bar{\tau}^{\rm tot}_{txy}\tilde{v}_t)\frac{\partial\gamma}{\partial x}-(\bar{\tau}_{lxy}\tilde{u}_l+\bar{\tau}_{lyy}\tilde{v}_l-\bar{\tau}^{\rm tot}_{txy}\tilde{u}_t-\bar{\tau}^{\rm tot}_{tyy}\tilde{v}_t)\frac{\partial\gamma}{\partial y}\\&+(\bar{q}_{lx}-\bar{q}^{\rm tot}_{tx})\frac{\partial\gamma}{\partial x}+(\bar{q}_{ly}-\bar{q}^{\rm tot}_{ty})\frac{\partial\gamma}{\partial y}.\end{aligned}$$

The mean total energy $\tilde{E}_t$ and mean total enthalpy $\tilde{H}_t$ during the turbulent phase are given by

$$\tilde{E}_t=\tilde{e}_t+\tfrac{1}{2}(\tilde{u}_t^2+\tilde{v}_t^2)+\tilde{k}_t,\qquad \tilde{H}_t=\tilde{E}_t+\frac{\bar{p}_t}{\bar{\rho}_t}+\tfrac{2}{3}\tilde{k}_t,$$

where $\tilde{e}_t$ is the mean internal energy. $\bar{\tau}^{\rm tot}_{tij}$ are stress components formed by the sum of the Reynolds stress components and the mean molecular stress components during the turbulent phase. In the same way $\bar{q}^{\rm tot}_{ti}$ are total heat flux components during the turbulent phase.

CONDITIONED TURBULENCE EQUATIONS

We derive here the equation for the turbulence kinetic energy during the turbulent phase, $\tilde{k}_t$. From a combination of the conditioned continuity equation and the conditioned momentum equations an equation for the mean flow kinetic energy during the turbulent phase can be derived. This equation is

$$\frac{\partial\bar{\rho}_t\frac{1}{2}\tilde{u}_{ti}^2}{\partial t}+\frac{\partial\bar{\rho}_t\frac{1}{2}\tilde{u}_{ti}^2\tilde{u}_{tj}}{\partial x_j}+\tilde{u}_{ti}\frac{\partial\bar{p}_t}{\partial x_i}-\tilde{u}_{ti}\frac{\partial\bar{\tau}^{\rm tot}_{tij}}{\partial x_j}=\tilde{u}_{ti}B_i-\tfrac{1}{2}\tilde{u}_{ti}^2A, \tag{17}$$

where A is the source term in the conditioned turbulent continuity equation (9) and B_i are the source terms in the conditioned turbulent momentum equations due to intermittency. The term B_x can be seen in equation (12) and is equal to $(1/2\gamma)S^{\gamma}_{\rho u}$. Equation (17) is similar to the equation for the global mean flow kinetic energy but differs from this equation by the source terms due to intermittency. From the unaveraged continuity equation and the momentum equations, similar to (17) the equation for the unaveraged kinetic energy is found as

$$\frac{\partial\rho\frac{1}{2}u_i^2}{\partial t}+\frac{\partial\rho\frac{1}{2}u_i^2u_j}{\partial x_j}+u_i\frac{\partial p}{\partial x_i}-u_i\frac{\partial\tau_{ij}}{\partial x_j}=0.$$

The conditioned averaged equation during the turbulent phase corresponding to (18) is

$$
\begin{aligned}
&\frac{\partial(\overline{\rho}_{\mathrm{t}}\frac{1}{2}\tilde{u}_{\mathrm{t}i}^2+\overline{\rho}_{\mathrm{t}}\tilde{k}_{\mathrm{t}})}{\partial t}+\frac{\partial(\overline{\rho}_{\mathrm{t}}\frac{1}{2}\tilde{u}_{\mathrm{t}i}^2\tilde{u}_{\mathrm{t}j}+\overline{\rho}_{\mathrm{t}}\tilde{k}_{\mathrm{t}}\tilde{u}_{\mathrm{t}j})}{\partial x_j}+\frac{\partial(\tilde{u}_{\mathrm{t}i}\overline{\rho u''_{\mathrm{t}i}u''_{\mathrm{t}j}}+\frac{1}{2}\overline{\rho u''^2_{\mathrm{t}i}u''_{\mathrm{t}j}})}{\partial x_j}\\
&\quad+\tilde{u}_{\mathrm{t}i}\frac{\partial\overline{p}_{\mathrm{t}}}{\partial x_i}+\overline{u''_{\mathrm{t}i}}\frac{\partial\overline{p}_{\mathrm{t}}}{\partial x_i}+\overline{u''_{\mathrm{t}i}\frac{\partial p'_{\mathrm{t}}}{\partial x_i}}-\tilde{u}_{\mathrm{t}i}\frac{\partial\overline{\tau_{\mathrm{t}ij}}}{\partial x_j}-\overline{u''_{\mathrm{t}i}}\frac{\partial\overline{\tau_{\mathrm{t}ij}}}{\partial x_j}-\overline{u''_{\mathrm{t}i}\frac{\partial\tau'_{\mathrm{t}ij}}{\partial x_j}}\\
&=\frac{1}{2\gamma}\left((\overline{\rho}_{\mathrm{l}}\tfrac{1}{2}\tilde{u}_{\mathrm{l}i}^2-\overline{\rho}_{\mathrm{t}}\tfrac{1}{2}\tilde{u}_{\mathrm{t}i}^2-\overline{\rho}_{\mathrm{t}}\tilde{k}_{\mathrm{t}})\frac{\partial\gamma}{\partial t}+(\overline{\rho}_{\mathrm{l}}\tfrac{1}{2}\tilde{u}_{\mathrm{l}i}^2\tilde{u}_{\mathrm{l}j}-\overline{\rho}_{\mathrm{t}}\tfrac{1}{2}\tilde{u}_{\mathrm{t}i}^2\tilde{u}_{\mathrm{t}j}-\overline{\rho}_{\mathrm{t}}\tilde{k}_{\mathrm{t}}\tilde{u}_{\mathrm{t}j}-\tilde{u}_{\mathrm{t}i}\overline{\rho u''_{\mathrm{t}i}u''_{\mathrm{t}j}}-\tfrac{1}{2}\overline{\rho u''^2_{\mathrm{t}i}u''_{\mathrm{t}j}})\frac{\partial\gamma}{\partial x_j}\right)\\
&\quad+\frac{1}{\gamma}\left((\tfrac{1}{3}\tilde{u}_{\mathrm{t}i}+\tfrac{1}{3}\overline{u''_{\mathrm{t}i}}+\tfrac{1}{6}\tilde{u}_{\mathrm{l}i})(\overline{p}_{\mathrm{l}}-\overline{p}_{\mathrm{t}})\frac{\partial\gamma}{\partial x_i}-\tfrac{1}{3}\overline{u''_{\mathrm{t}i}p'_{\mathrm{t}}}\frac{\partial\gamma}{\partial x_i}-(\tfrac{1}{3}\tilde{u}_{\mathrm{t}i}+\tfrac{1}{3}\overline{u''_{\mathrm{t}i}}+\tfrac{1}{6}\tilde{u}_{\mathrm{l}i})(\overline{\tau_{\mathrm{l}ij}}-\overline{\tau_{\mathrm{t}ij}})\frac{\partial\gamma}{\partial x_j}+\tfrac{1}{3}\overline{u''_{\mathrm{t}i}\tau'_{\mathrm{t}ij}}\frac{\partial\gamma}{\partial x_j}\right).
\end{aligned}
\tag{19}
$$

We denote the source term due to intermittency in (19) by C. By combining (17) and (19), the equation for the turbulence kinetic energy during the turbulent phase is found as

$$
\begin{aligned}
&\frac{\partial\overline{\rho}_{\mathrm{t}}\tilde{k}_{\mathrm{t}}}{\partial t}+\frac{\partial\overline{\rho}_{\mathrm{t}}\tilde{k}_{\mathrm{t}}\tilde{u}_{\mathrm{t}j}}{\partial x_j}+\frac{\partial(-\tilde{u}_{\mathrm{t}i}\overline{\tau}^{\mathrm{R}}_{ij}+\frac{1}{2}\overline{\rho u''^2_{\mathrm{t}i}u''_{\mathrm{t}j}})}{\partial x_j}+\overline{u''_{\mathrm{t}i}}\frac{\partial\overline{p}_{\mathrm{t}}}{\partial x_i}+\overline{u''_{\mathrm{t}i}\frac{\partial p'_{\mathrm{t}}}{\partial x_i}}+\tilde{u}_{\mathrm{t}i}\frac{\partial\overline{\tau}^{\mathrm{R}}_{ij}}{\partial x_j}-\overline{u''_{\mathrm{t}i}}\frac{\partial\overline{\tau}_{\mathrm{t}ij}}{\partial x_j}-\overline{u''_{\mathrm{t}i}\frac{\partial\tau'_{\mathrm{t}ij}}{\partial x_j}}\\
&\qquad=C-\tilde{u}_{\mathrm{t}i}B_i+\tfrac{1}{2}\tilde{u}_{\mathrm{t}i}^2A.
\end{aligned}
\tag{20}
$$

The left hand side of this turbulence kinetic energy equation has the same form as the global turbulence kinetic energy equation. We recognize the following terms:

production $\quad P_k=\overline{\tau}^{\mathrm{R}}_{ij}\dfrac{\partial\tilde{u}_{\mathrm{t}i}}{\partial x_j}$,

diffusion $\quad \dfrac{\partial(\overline{u''_{\mathrm{t}i}\tau'_{\mathrm{t}ij}}-\overline{u''_{\mathrm{t}j}p'_{\mathrm{t}}}-\frac{1}{2}\overline{\rho u''^2_{\mathrm{t}i}u''_{\mathrm{t}j}})}{\partial x_j}$,

dissipation $\quad -\overline{\tau'_{\mathrm{t}ij}\dfrac{\partial u''_{\mathrm{t}i}}{\partial x_j}}$,

compressibility $\quad \overline{p'_{\mathrm{t}}\dfrac{\partial u''_{\mathrm{t}j}}{\partial x_j}}-\overline{u''_{\mathrm{t}i}}\dfrac{\partial\overline{p}_{\mathrm{t}}}{\partial x_i}+\overline{u''_{\mathrm{t}i}}\dfrac{\partial\overline{\tau_{\mathrm{t}ij}}}{\partial x_j}$.

The source term due to intermittency in the $\tilde{k}_{\mathrm{t}}$-equation can be worked out into

$$
\begin{aligned}
C-\tilde{u}_{\mathrm{t}i}B_i+\tfrac{1}{2}\tilde{u}_{\mathrm{t}i}^2A&=[\tfrac{1}{2}\overline{\rho}_{\mathrm{l}}(\tilde{u}_{\mathrm{l}i}-\tilde{u}_{\mathrm{t}i})^2-\overline{\rho}_{\mathrm{t}}\tilde{k}_{\mathrm{t}}]\frac{1}{2\gamma}\frac{\partial\gamma}{\partial t}\\
&\quad+[\tfrac{1}{2}\overline{\rho}_{\mathrm{l}}(\tilde{u}_{\mathrm{l}i}-\tilde{u}_{\mathrm{t}i})^2\tilde{u}_{\mathrm{l}j}-\tfrac{1}{3}\overline{\rho}_{\mathrm{t}}\tilde{k}_{\mathrm{t}}\tilde{u}_{\mathrm{t}j}-\tfrac{1}{2}\overline{\rho u''^2_{\mathrm{t}i}u''_{\mathrm{t}j}}]\frac{1}{2\gamma}\frac{\partial\gamma}{\partial x_j}\\
&\quad+[(\tfrac{1}{3}\overline{u''_{\mathrm{t}i}}+\tfrac{1}{6}\tilde{u}_{\mathrm{l}i}-\tfrac{2}{3}\tilde{u}_{\mathrm{t}i})(\overline{p}_{\mathrm{l}}-\overline{p}_{\mathrm{t}})-\tfrac{1}{3}\overline{u''_{\mathrm{t}i}p'_{\mathrm{t}}}]\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_i}\\
&\quad-[(\tfrac{1}{3}\overline{u''_{\mathrm{t}i}}+\tfrac{1}{6}\tilde{u}_{\mathrm{l}i}-\tfrac{2}{3}\tilde{u}_{\mathrm{t}i})(\overline{\tau_{\mathrm{l}ij}}-\overline{\tau_{\mathrm{t}ij}})-\tfrac{1}{3}\overline{u''_{\mathrm{t}i}\tau'_{\mathrm{t}ij}}]\frac{1}{\gamma}\frac{\partial\gamma}{\partial x_j}.
\end{aligned}
\tag{21}
$$

In each of the four parts in this source term the components which are grouped into square brackets more or less compensate for each other. We consider as an example the first group of components. In a wall-bounded flow the turbulent mean velocity component in the direction of the wall is larger than the laminar velocity component near the wall, while the reverse is true far away from the wall. Thus near to the wall the considered coefficient is positive, while further away from the wall, where laminar and turbulent velocities are approximately equal, the coefficient is negative. Therefore the mean influence on the generation of turbulence kinetic energy is very low. The same can be said from the other three parts. The conclusion is that the source term in the $\tilde{k}_t$-equation has only a second-order effect, in the sense that it can alter the distribution of $\tilde{k}_t$ in a wall-bounded flow, but not the mean level. Therefore taking into account the modelling which anyhow has to be done on the left hand side of equation (20), it seems appropriate to neglect the right-hand side. We verified the influence of the source term by including the first and second groups of terms but leaving out $\frac{1}{2}\overline{\rho u''^2_{ti} u''_{tj}}$ in the calculation. These introduced terms are free of any modelling. The influence on the results was completely negligible. The source terms in the Navier–Stokes equation (9)–(16) are much more significant and cannot be deleted.

It is very important to come to the conclusion that the source terms in the turbulence equations can be neglected. For the $\tilde{k}_t$-equation it would not be difficult to introduce models for the terms in (21) that need closure, since these terms are linked to the diffusion process. It would, however, be almost impossible to construct the second equation for turbulence quantities since, for instance, the ϵ-equation has not at all the same rational basis as the k-equation.

We model the turbulence by the classical (low-Reynolds-number) k- and ϵ-equations, but written for the turbulent conditioned averaged values. Owing to the good representations for zero (ZPG), favourable (FPG) and adverse (APG) pressure gradient flows, the Yang–Shih variant[18] has been chosen as the low-Reynolds-number turbulence model. The equations are

$$\frac{\partial \bar{\rho}_t \tilde{k}_t}{\partial t} + \frac{\partial \bar{\rho}_t \tilde{k}_t \tilde{u}_t}{\partial x} + \frac{\partial \bar{\rho}_t \tilde{k}_t \tilde{v}_t}{\partial y} = \frac{\partial}{\partial x}\left[\left(\bar{\mu}_t + \frac{\mu_e}{\sigma_k}\right)\frac{\partial \tilde{k}_t}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(\bar{\mu}_t + \frac{\mu_e}{\sigma_k}\right)\frac{\partial \tilde{k}_t}{\partial y}\right] + \tilde{P}_k - \bar{\rho}_t \tilde{\epsilon}_t,$$

$$\frac{\partial \bar{\rho}_t \tilde{\epsilon}_t}{\partial t} + \frac{\partial \bar{\rho}_t \tilde{\epsilon}_t \tilde{u}_t}{\partial x} + \frac{\partial \bar{\rho}_t \tilde{\epsilon}_t \tilde{v}_t}{\partial y} = \frac{\partial}{\partial x}\left[\left(\bar{\mu}_t + \frac{\mu_e}{\sigma_\epsilon}\right)\frac{\partial \tilde{\epsilon}_t}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(\bar{\mu}_t + \frac{\mu_e}{\sigma_\epsilon}\right)\frac{\partial \tilde{\epsilon}_t}{\partial y}\right] + (C_{\epsilon 1}\tilde{P}_k - C_{\epsilon 2} f_2 \bar{\rho}_t \tilde{\epsilon})\frac{1}{\mathscr{T}} + \mathscr{E},$$

where

$$\tilde{P}_k = \left[\mu_e\left(\frac{\partial \tilde{u}_{ti}}{\partial x_j} + \frac{\partial \tilde{u}_{tj}}{\partial x_i} - \tfrac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_{tk}}{\partial x_k}\right) - \tfrac{2}{3}\delta_{ij}\bar{\rho}_t \tilde{k}_t\right]\frac{\partial \tilde{u}_{ti}}{\partial x_j},$$

$$\mu_e = C_\mu f_\mu \bar{\rho}_t \tilde{k}_t \mathscr{T}, \qquad \mathscr{T} = \frac{\tilde{k}_t}{\tilde{\epsilon}_t} + \sqrt{\left(\frac{\mu}{\bar{\rho}_t \tilde{\epsilon}_t}\right)}, \qquad \mathscr{E} = \frac{\bar{\mu}_t \mu_e}{\bar{\rho}_t}\frac{\partial^2 \tilde{u}_{ti}}{\partial x_j \partial x_k}\frac{\partial^2 \tilde{u}_{ti}}{\partial x_j \partial x_k},$$

$$f_\mu = 1 - \exp(a_1 R_y + a_3 R_y^3 + a_5 R_y^5), \qquad R_y = \frac{\bar{\rho}_t \sqrt{k_t} y}{\mu},$$

$$f_2 = 1 - \exp\left(-\frac{R_t^2}{36}\right), \qquad R_t = \frac{\bar{\rho}_t \tilde{k}_t \mathscr{T}}{\bar{\mu}_t}.$$

The following model constants are used: $C_{\epsilon 1} = 1{\cdot}44$, $C_{\epsilon 2} = 1{\cdot}92$, $\sigma_k = 1$, $\sigma_\epsilon = 1{\cdot}3$, $C_\mu = 0{\cdot}09$, $a_1 = -1{\cdot}5 \times 10^{-4}$, $a_3 = -5 \times 10^{-7}$, $a_5 = -1 \times 10^{-10}$. At the wall, $\tilde{k}_t = 0$ and $\bar{\rho}_t \tilde{\epsilon}_{tw} = 2\bar{\mu}_t(\partial\sqrt{\tilde{k}_t}/\partial y)^2$ are imposed.

INTERMITTENCY MODELLING

For zero pressure gradient (ZPG) the intermittency γ can be described algebraically according to Dhawan and Narasimha[12] by

$$\gamma(x) = 1 - \exp[-\hat{n}\sigma(Re_x - Re_{x\mathrm{tr}})^2], \tag{22}$$

where $\hat{n}$ is the non-dimensional turbulence spot production rate and σ the turbulent spot propagation parameter. This law is valid for concentrated breakdown at x_{tr}, which is typical of natural transition. Concentrated breakdown can be recognized in the linear law of the function $F(\gamma) = \sqrt{[-\ln(1-\gamma)]}$ (broken line in Figure 4) starting from x_{tr}.

According to Mayle[19] and Gostelow and Walker,[20] in bypass transition a Gaussian distribution of the spot production at the onset of transition is more realistic. To take account of this distributed breakdown, the growth parameter $\hat{n}\sigma$ cannot be seen as constant in the beginning of the transition zone. Figure 4 shows schematically the evolution of $F(\gamma)$ as a function of distance for distributed breakdown.[20] To evaluate γ, we first derive from (22) by differentiating:

$$\frac{1}{1-\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}x} = 2A(x - x_{\mathrm{tr}}), \qquad A = \hat{n}\sigma\frac{U^2}{\nu^2}. \tag{23}$$

Generally, formula (23) and $F(\gamma)$ can be written as

$$\frac{1}{1-\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}x} = \beta(x), \qquad F(\gamma) = f(x).$$

Both expressions lead to a γ-function:

$$\gamma = 1 - \exp\left(-\int_{x_{\mathrm{tr}}}^{x} \beta(x)\mathrm{d}x\right) \quad \text{and} \quad \gamma = 1 - \exp[-f^2(x)].$$

As both distributions must be the same, this results in

$$\beta(x) = 2f(x)f'(x).$$

Thus any $F(\gamma)$-distribution can be used to evaluate γ by

$$\frac{\mathrm{d}\gamma}{\mathrm{d}x} = (1-\gamma)2f(x)f'(x). \tag{24}$$

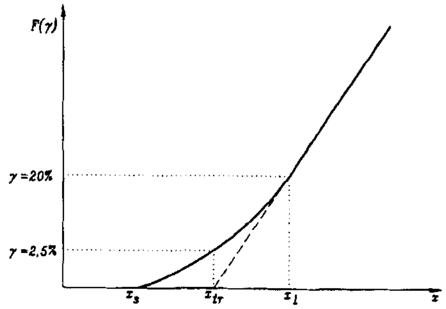

Figure 4. Intermittency in distributed breakdown

In the case of a distributed breakdown the linear growth is obtained at about the level $\gamma = 20\%$. To model the initial behaviour of the growth, we draw a rational function, with $x' = x - x_s$, of the form

$$f(x) = \frac{ax'^4 + bx'^3 + cx'^2 + dx' + e}{x'^3 + f}$$

through the points x_{st}, x_{tr} and x_{li} with γ-levels 1%, 2·5% and 20%, with $x_{li} - x_{tr} = x_{tr} - x_{st}$ and boundary conditions $d\gamma/dx = 0$ at $x = x_{st}$ and $d\gamma/dx$ equal to the value of the linear law at $x = x_{li}$. To be sure that the function fits the linear law further away, an extra point is considered for $\gamma = 80\%$. The corresponding values are $a = \sqrt{A}, b = -0{\cdot}4906, c = 0{\cdot}204A^{-0{\cdot}5}, d = 0, e = 0{\cdot}04444A^{-1{\cdot}5}$ and $f = 10e$.

For bypass transition the parameter $\hat{n}\sigma$ and the point of transition, x_{tr}, have been correlated by Mayle[19] based on intermittency measurements for zero pressure gradient flow as

$$\hat{n}\sigma = 1{\cdot}25 \times 10^{-11} Tu^{7/4}, \tag{25}$$

$$Re_{\theta tr} = 420\ Tu^{-0{\cdot}69}, \tag{26}$$

where Tu-intensity is in %. In the experimental results used by Mayle to derive these correlations, there is always uncertainty about the location where the turbulence level is defined. We take here the value at the leading edge.

To the best of our knowledge there are no criteria available for the start of transition with distributed breakdown. Therefore, in the calculations we use formula (26) but applied to the start of transition x_{st}, instead of x_{tr}.

For non-parallel flow we can assume that (24) holds along a streamline:

$$\bar{\rho}\tilde{u}_s \frac{\partial \gamma}{\partial s} = (1 - \gamma)\beta(s)\bar{\rho}\tilde{u}_s, \tag{27}$$

where $\tilde{u}_s$ is the modulus of the global velocity and s is the co-ordinate along the streamline. In a Cartesian co-ordinate system (27) can be written as

$$\bar{\rho}\tilde{u}\frac{\partial \gamma}{\partial x} + \bar{\rho}\tilde{v}\frac{\partial \gamma}{\partial y} = (1 - \gamma)\beta(s)\bar{\rho}\sqrt{(\tilde{u}^2 + \tilde{v}^2)}. \tag{28}$$

The value of s can be determined by

$$s = \int_{st} \frac{\tilde{u}\,dx + \tilde{v}\,dy}{\sqrt{(\tilde{u}^2 + \tilde{v}^2)}}.$$

Finally, for transition in a non-ZPG a pressure correlation is needed accounting for the difference in spot growth. Figure 5 gives the spot growth parameter normalized by the flat plate correlation (25) for a whole range of acceleration parameters ($K = (\nu/U^2)\,dU/dx$) and turbulence levels (Tu in %). The data for the adverse pressure gradients are derived from the experiments of Gostelow *et al.*[4], while the data for $K > 0$ are derived from Blair's data.[3] By fitting the law (22) to the central part of the experimentally determined γ-distribution, the spot growth parameter is determined. The following correlations are proposed:

$$\frac{\hat{n}\sigma}{\hat{n}\sigma_{FP}} = \begin{cases} (474Tu^{-2{\cdot}9})^{1-\exp(2\times 10^6 K)}, & K < 0, \\ 10^{-3227K^{0{\cdot}5985}}, & K > 0 \end{cases}. \tag{29}$$

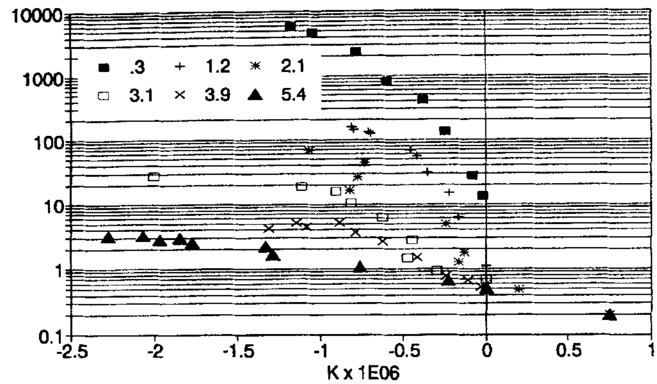

Figure 5. Normalized turbulent spot production rate as a function of acceleration parameter K at different turbulence levels

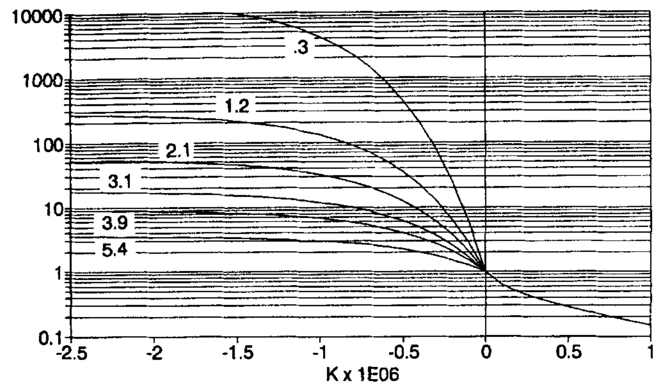

Figure 6. Correlations for normalized turbulent spot production rate as a function of acceleration parameter K at different turbulence levels

The correlations are drawn in Figure 6 for different Tu-levels as a function of the acceleration parameter K. The ratio in Figure 6, for $K > 0$ is independent of the turbulence level. For $K < 0$, the ratio depends on the turbulence level and becomes independent of the acceleration parameter K for strongly decelerating flows.

NUMERICAL PROCEDURE

The conditionally averaged laminar and turbulent equations together with the intermittency transport equation can be written in vector form as

$$\frac{\partial \overline{U}_{\mathrm{l}}}{\partial t} + \frac{\partial \overline{F}_{\mathrm{l}}}{\partial x} + \frac{\partial \overline{G}_{\mathrm{l}}}{\partial y} = \frac{\partial \overline{F}_{v\mathrm{l}}}{\partial x} + \frac{\partial \overline{G}_{v\mathrm{l}}}{\partial y} + S_{\mathrm{l}}^{\gamma}, \tag{30}$$

$$\frac{\partial \overline{U}_{\mathrm{t}}}{\partial t} + \frac{\partial \overline{F}_{\mathrm{t}}}{\partial x} + \frac{\partial \overline{G}_{\mathrm{t}}}{\partial y} = \frac{\partial \overline{F}_{v\mathrm{t}}}{\partial x} + \partial \frac{\overline{G}_{v\mathrm{t}}}{\partial y} + S_{\mathrm{t}}^{\gamma}, \tag{31}$$

$$\frac{\partial \overline{\rho}\gamma}{\partial t} + \frac{\partial \overline{\rho}\overline{u}\gamma}{\partial x} + \frac{\partial \overline{\rho}\overline{v}\gamma}{\partial y} = S_{\gamma}. \tag{32}$$

Equation (30) describes the laminar phase and differs from the classical Navier–Stokes equations by the presence of the source terms

$$S_l^{\gamma^T} = \frac{1}{2(1-\gamma)}\{S_\rho^\gamma; S_{\rho u}^\gamma; S_{\rho v}^\gamma; S_{\rho E}^\gamma\}.$$

The second set of equations describes the turbulent phase and consists of the four Navier–Stokes equations along with the turbulent k–ϵ equations. Except for the turbulence equations, there are also source terms present for the Navier-Stokes part. These source terms can be written concisely as

$$S_t^{\gamma^T} = \left\{\frac{1}{2\gamma}S_\rho^\gamma; \frac{1}{2\gamma}S_{\rho u}^\gamma; \frac{1}{2\gamma}S_{\rho v}^\gamma; \frac{1}{2\gamma}S_{\rho E}^\gamma; S_{\rho k}; S_{\rho\epsilon}\right\}.$$

The intermittency equation is activated at the transition point x_{st} which corresponds with $\gamma = 0{\cdot}01$. For $x < x_{st}$ the intermittency factor γ is set at the 1% value. The upper value of γ is restricted to 99% in order to avoid singularities.

The integration of the laminar, the turbulent and the intermittency equation is based upon an upwind discretization. Using first-order upwind differencing guarantees the positiviness of the system so that it can be solved by any relaxation scheme. The second order correction to the fluxes is constructed by the flux extrapolation technique involving a minmod limiter. This contribution has no definite character and is put into a correction cycle. Full details on the splitting and the second-order correction are given in References 21 and 22. In Reference 22 special attention is given to the treatment of the source terms originating from the k–ϵ equations. In order to assure the positivity of the system, the negative source terms are linearized and put into the left hand side. The same technique is applied on the negative parts of the source terms in the conditioned Navier–Stokes equations. No special linearization is necessary for the intermittency equation as the source term S_γ is always positive.

RESULTS

In earlier work[23,24] the intermittency factor γ was prescribed algebraically according to the law of Dhawan and Narasimha,[12] equation (22). The necessity of bringing the Gaussian distribution into the spot production was recognized very soon. In the following we discuss three different test cases: flows with a zero, a favourable and an adverse pressure gradient.

The specifications of the different test cases are given in Table I, where U_∞ stands for the velocity upstream of the leading edge (LE), Tu_{le} is the turbulence level at the LE, $x_{grid} - x_{le}$ is the relative position of the turbulence generating grid w.r.t. the LE and $\overline{K}$ is the mean acceleration parameter within the transition zone. The first test case, CU, comes from Kuan and Wang of Clemson University and is described in detail in Reference 25. The T3C1 case is a combined favourable and adverse pressure gradient flow typical of an aft-loaded turbine. This test case was proposed by ERCOFTAC. The velocity distribution along the flat plate with sharp leading edge is given in Figure 7(a). The final case,

Table I: Description of different test cases

Case	U_∞ (m s^{-1})	Tu_{le} (%)	$x_{grid} - x_{le}$ (mm)	$\overline{K}$
CU	13·8	1·1	900	0
T3C1	6·12	7·78	610	$1{\cdot}75 \times 10^{-6}$
SUG5K6	15·28	3·9	1200	$-0{\cdot}9 \times 10^{-6}$

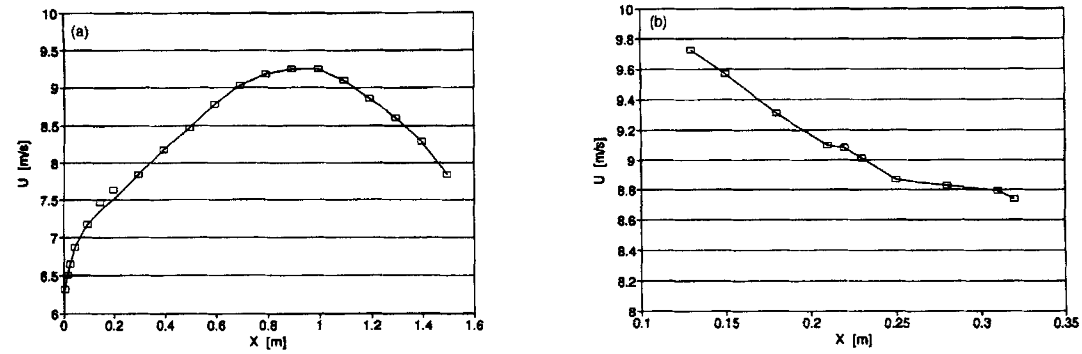

Figure 7. Velocity distribution along flat plate for (a) T3C1 and (b) SUG5K6

SUG5K6, has been tested by Gostelow *et al.* and is described extensively in References 4 and 26. The adverse pressure gradient acts on a flat plate with an rounded elliptical nose in order to prevent separation bubbles. Figure 7(b) gives the corresponding velocity distribution.

Zero pressure gradient (Case CU)

In Figure 8, the skin friction C_f, the shape factor H and the Reynolds number based on momentum thickness Re_θ, are given as functions of Re_x. The experimental values are represented by open rectangles. The lower line in the C_f-plot represents the exact Blasius-solution given by $C_f = 0{\cdot}664/\sqrt{Re_x}$. The upper line represents the relation for the turbulent skin friction: $C_f = 0{\cdot}445/\ln^2(0{\cdot}06Re_x)$. The predicted C_f-evolution is given by the third line. It is seen that both the transition point and the transition length are very well reproduced. Although not very clear, the experimental evolution of the shape factor starts already to deviate upstream of the transition point from the laminar value 2·59. This effect is, as will be seen later, more clearly present at higher turbulence levels. Through diffusion of turbulent eddies from the main flow towards the wall, the outer region of the boundary layer is affected first. As a consequence, the velocity profile in the outer region tends to a turbulent one through the presence of turbulent Reynolds stresses. This results in a decrease of the shape factor already upstream of the transition point. This diffusion should be taken into account in the modelling by defining a normal γ-distribution. This has not yet been introduced in the model.

Figure 9 shows the evolution of the profile of the global streamwise velocity fluctuation u' during transition for different positions along the plate. The global streamwise Reynolds normal stress is, with the usual approximation, given by (2):

$$\widetilde{u'^2} \approx \tilde{k} = \gamma\tilde{k}_t + \tfrac{1}{2}\gamma(1-\gamma)[(\tilde{u}_t - \tilde{u}_l)^2 + (\tilde{v}_t - \tilde{v}_l)^2], \tag{33}$$

where $\tilde{k}_t$ is the turbulence kinetic energy during the turbulent phase. The term $(1-\gamma)\tilde{k}_l$ is dropped in expression (33) as the laminar fluctuations are neglected in the calculation. At the start of transition the experimental data already show appreciable levels of $u' = \sqrt{\widetilde{u'^2}}$. The numerically predicted level is much lower owing to the neglect of laminar fluctuations. The laminar contribution to u' is important since it is multiplied by $1-\gamma$. Further in the transition phase the peak is well represented and corresponds well with the experimental data. Profiles of mean streamwise velocity components are shown in Figure 10. The correspondence with the experiments is quite reasonable. The evolution of the intermittency factor γ is shown in Figure 11. No experimental data are given at the start of transition since in the experiments no plateau was detected.

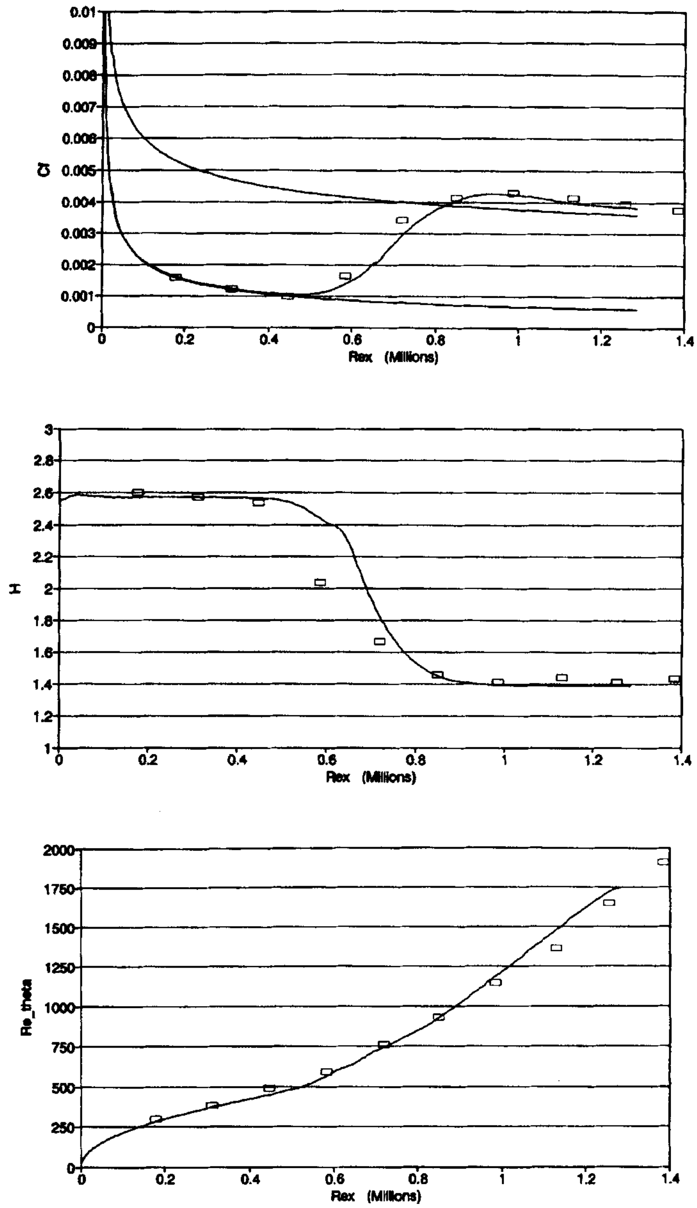

Figure 8. Skin friction coefficient (top), shape factor (middle) and Reynolds number based on momentum thickness (bottom) for CU (□, experimental values)

Favourable pressure gradient (Case T3C1)

The skin friction C_f, the shape factor H and the Reynolds number based on momentum thickness Re_θ are given in Figure 12. The start and length of transition are well reproduced in the C_f-plot. The lower and upper lines represent respectively the fully laminar and turbulent skin friction on a flat plate. As explained in the previous subsection, a larger discrepancy is present for the shape factor owing to

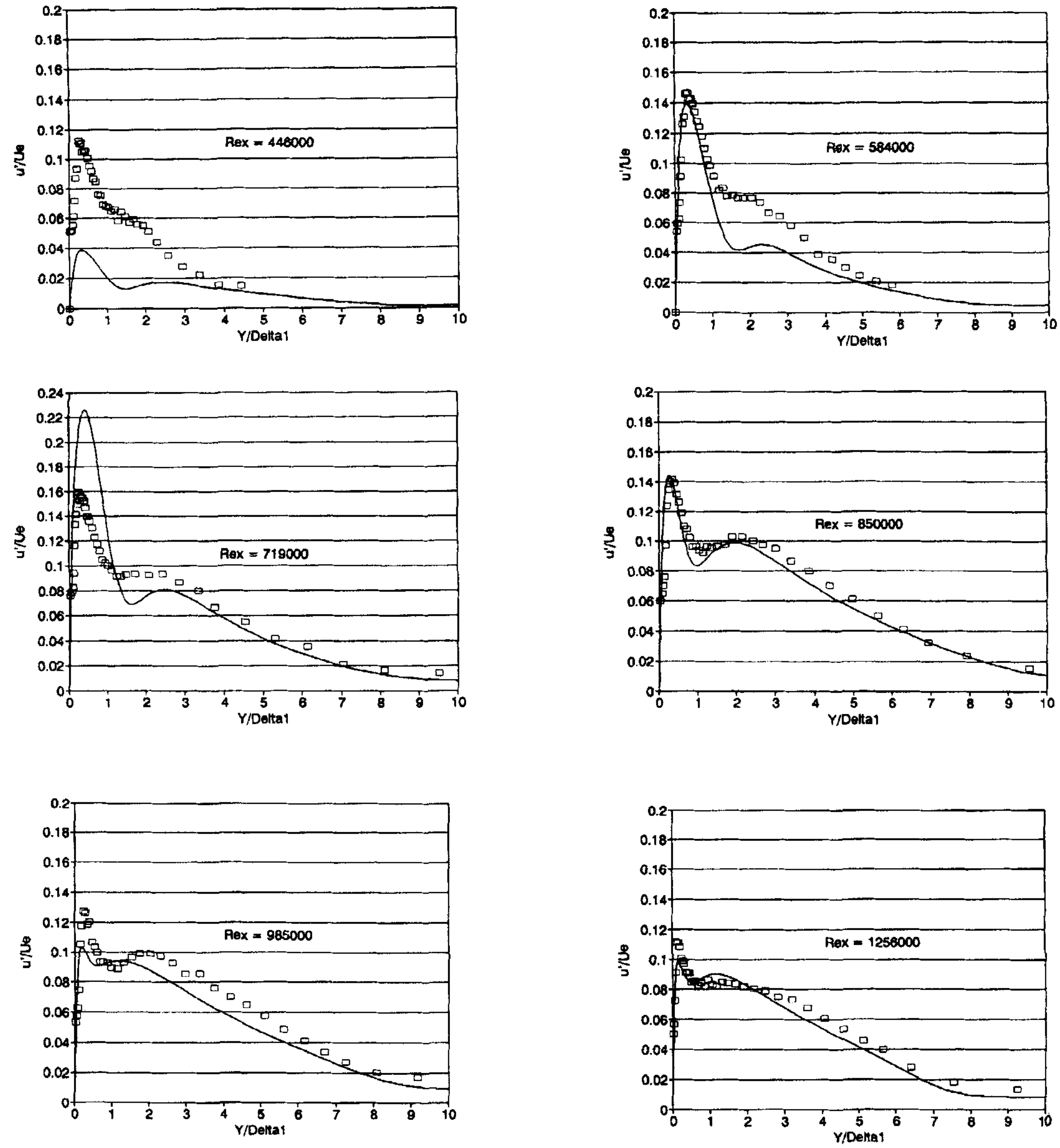

Figure 9. Profiles of fluctuating streamwise velocity component at six locations along plate for CU (□, experimental values)

the higher freestream turbulence. Figure 13 shows the evolution of the profile of the global streamwise velocity fluctuation u' during transition. Concerning the streamwise fluctuation, the same remarks can be made as in the CU-case: the fluctuations are underpredicted in the beginning of the intermittency zone while the levels correspond better further downstream. The mean velocity profiles, shown in Figure 14, correspond extremely well with the data. No experimental data for the intermittency factor γ are available for this test case.

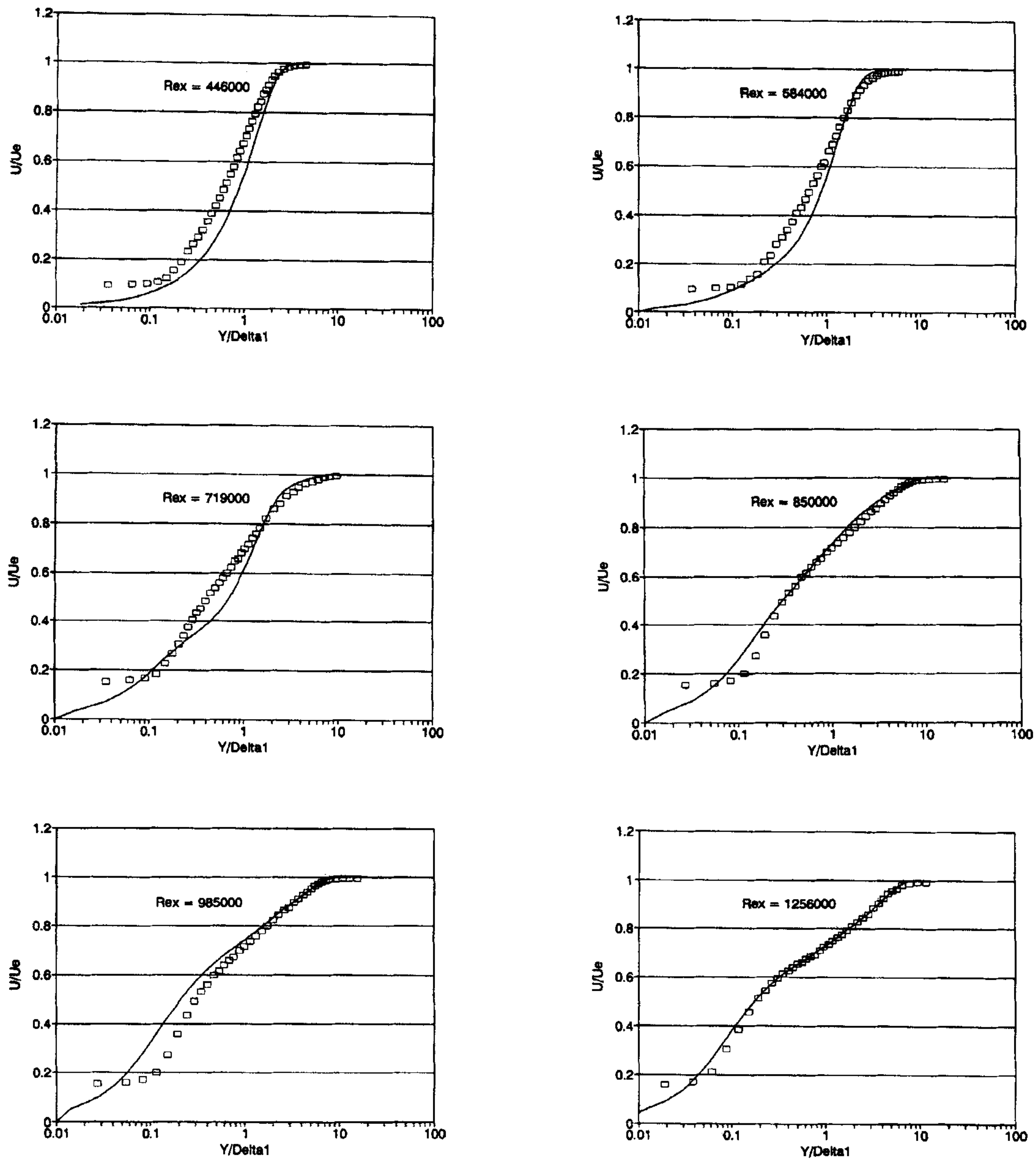

Figure 10. Mean velocity profiles at six locations along plate for CU (□, experimental values)

Adverse pressure gradient (Case SUG5K6)

Figure 15 gives the skin friction C_f, the shape factor H and the Reynolds number based on momentum thickness Re_θ. Unfortunately, no experimental data are available for the skin friction C_f. The predicted C_f can only be compared with the exact laminar (lower line) and turbulent (upper line) skin friction. Owing to the adverse pressure gradient and the high turbulence level, a larger discrepancy is present for the shape factor than in the previous cases. Both the pressure gradient and the turbulence

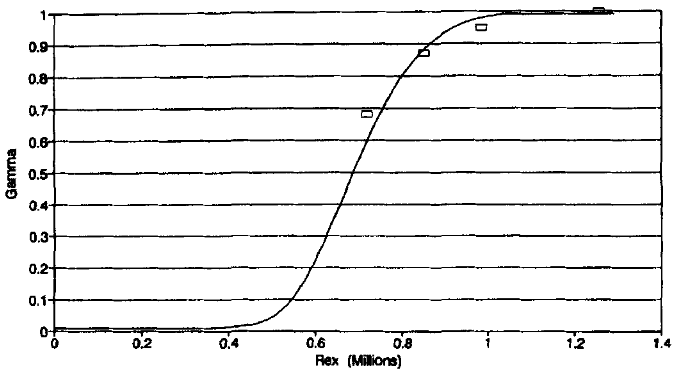

Figure 11. Evolution of intermittency factor γ along plate for CU (□, experimental values)

intensity enhance the diffusion of the turbulence towards the wall. At the leading edge both the skin friction and the shape factor show some oscillatory behaviour. This can be attributed to an irregularity of the leading edge shape in the experiments. We preferred to leave this irregularity in the calculations rather than to smooth it, because results are very sensitive to details in the shape. Globally, the momentum thickness has the same behaviour as in the measurements. As no experimental data are available, the distributions of the streamwise fluctuations are not given. The numerical results reveal peak values of 25% and higher. The mean velocity profiles, shown in Figure 16, correspond reasonably well with the data. The evolution of the intermittency factor γ is shown in Figure 17. The correspondence between the predicted evolution and the experimental data is quite good.

CONCLUSIONS

Conditioned averaged Navier–Stokes equations have been derived to model the transition zone. The source terms in the conditioned equations express the interaction between turbulent and non-turbulent parts in the flow. A transport equation for the intermittency factor has been derived. The source term in this equation represents the growth rate of the turbulent spots and is made dependent upon the acceleration parameter and the turbulence level. Introducing a dynamically determined intermittency factor leads to a very good prediction of the transitional behaviour. Skin friction and mean velocity profiles are in good agreement with measured profiles. The shape factor in the beginning of the transition deviates from the experimental values owing to the neglect of the normal variation of the intermittency factor. Except for the beginning of the transition, the turbulence profiles are well predicted. The discrepancy is due to the neglect of the laminar fluctuations.

ACKNOWLEDGEMENTS

The research reported here was granted under contract 9.0001.91 by the Belgian National Science Foundation (NFWO) and under contract IUAP/17 as part of the Belgian National Programme on Inter-University Poles of Attraction, initiated by the Federal Services of Scientific, Technical and Cultural Affairs (DWTC).

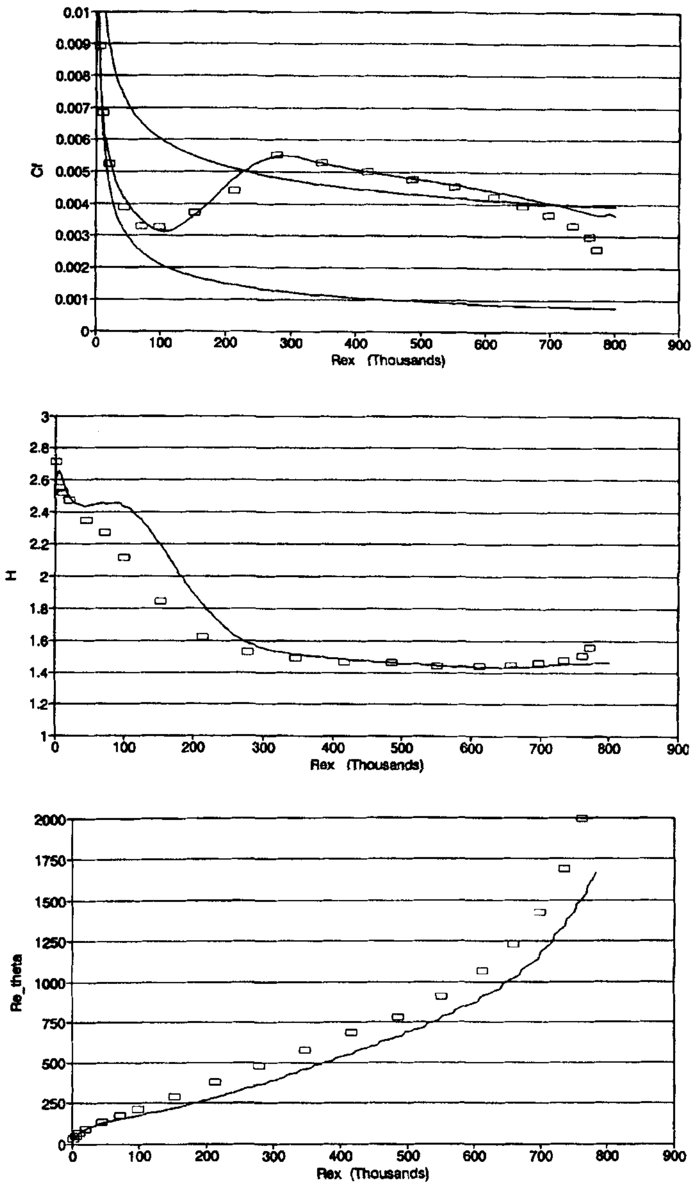

Figure 12. Skin friction coefficient (top), shape factor (middle) and Reynolds numbers based on momentum thickness (bottom) for T3C1 (□, experimental values)

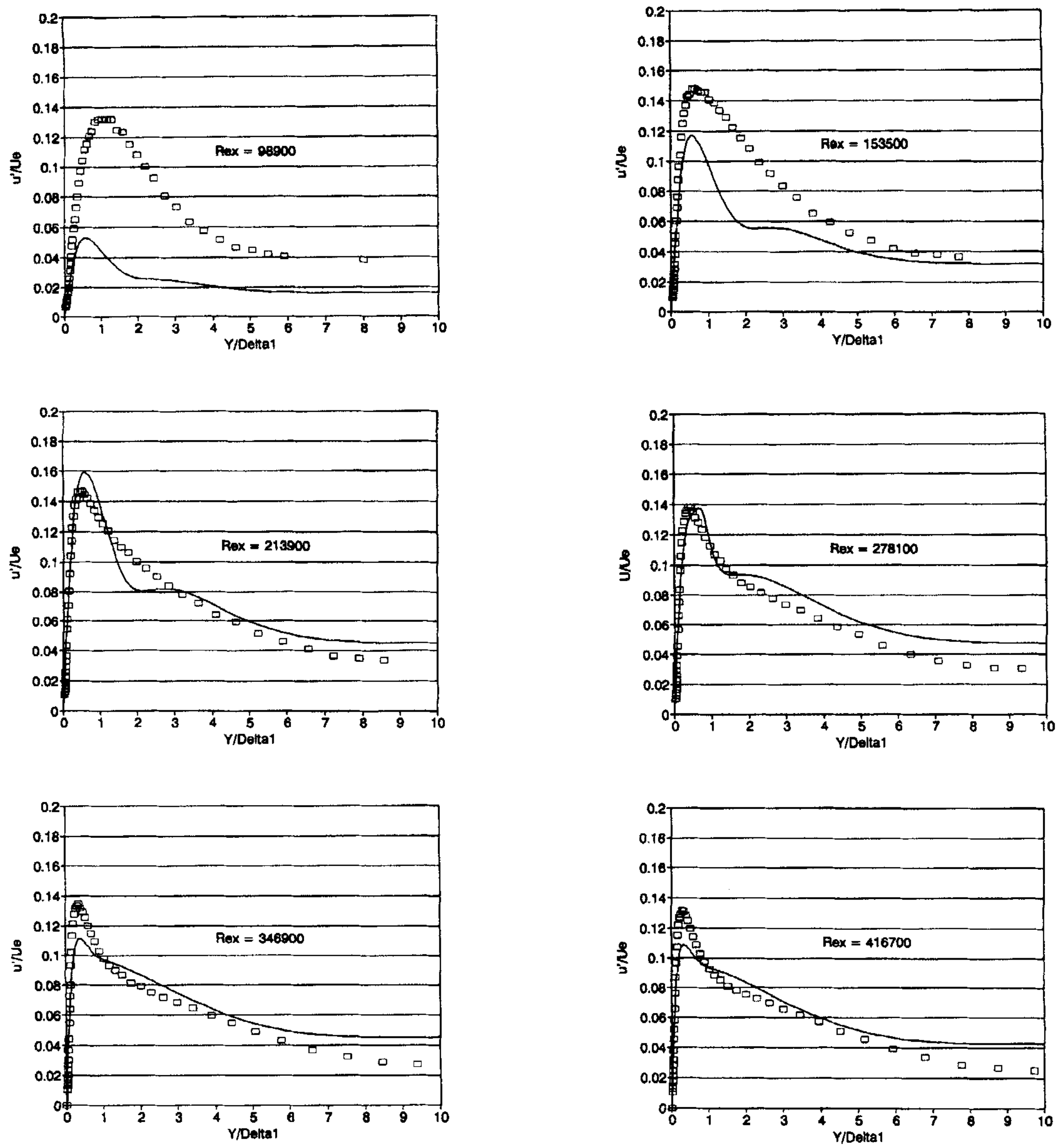

Figure 13. Profiles for fluctuating streamwise velocity component at six locations along plate for T3C1 (□, experimental values)

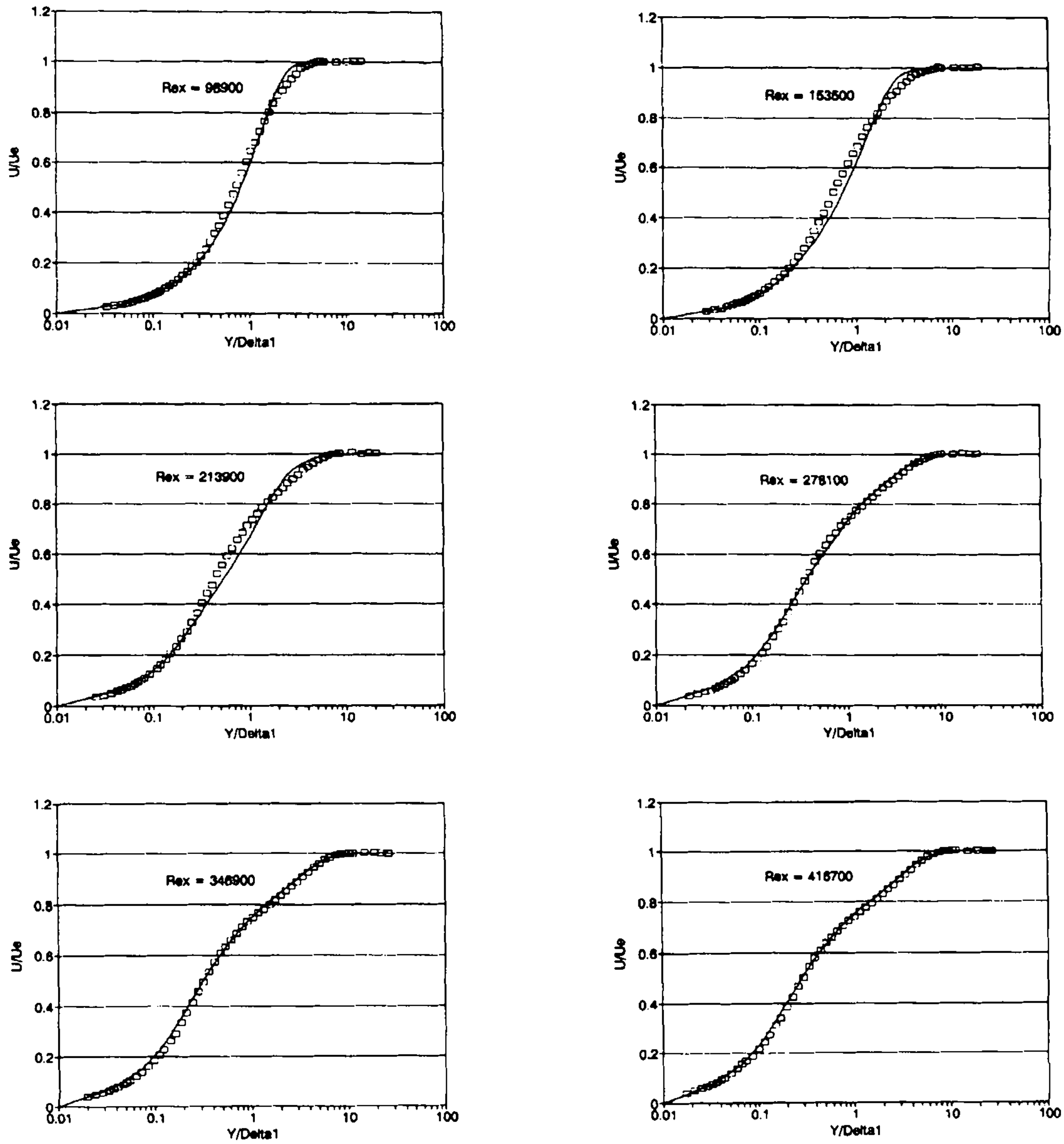

Figure 14. Mean velocity profiles at six locations along plate for T3C1 (□, experimental values)

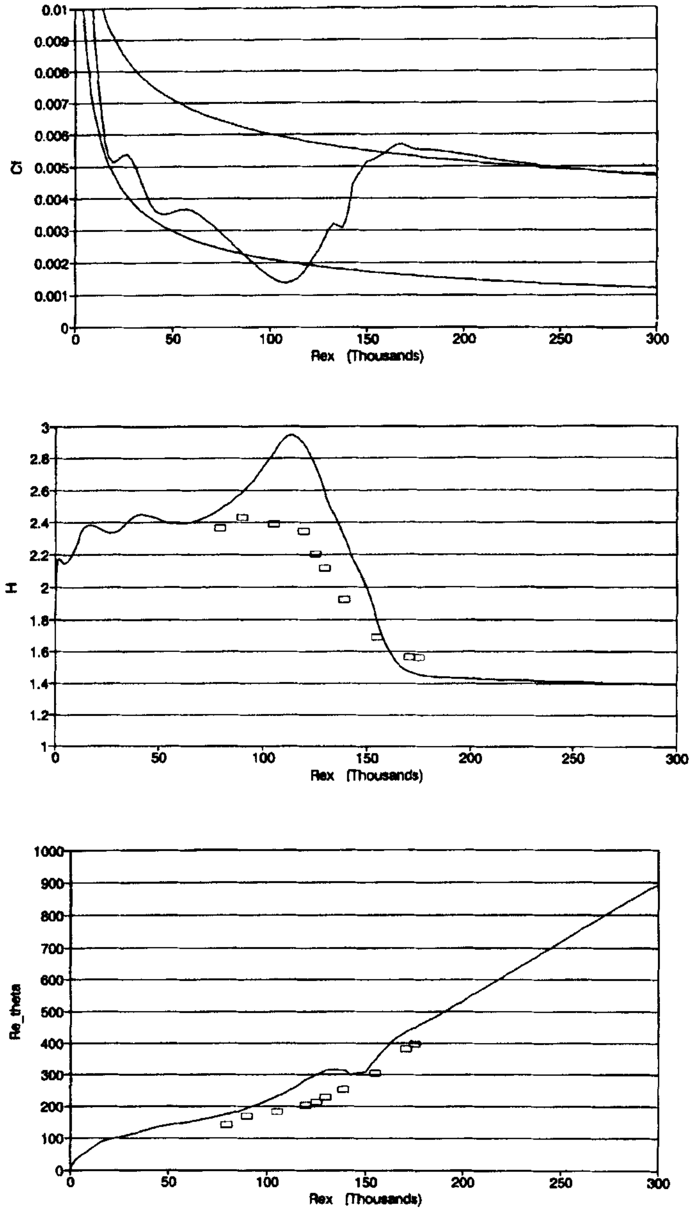

Figure 15. Skin friction coefficient (top), shape factor (middle) and Reynolds number based on momentum thickness (bottom) for SUG5K6 (□, experimental values)

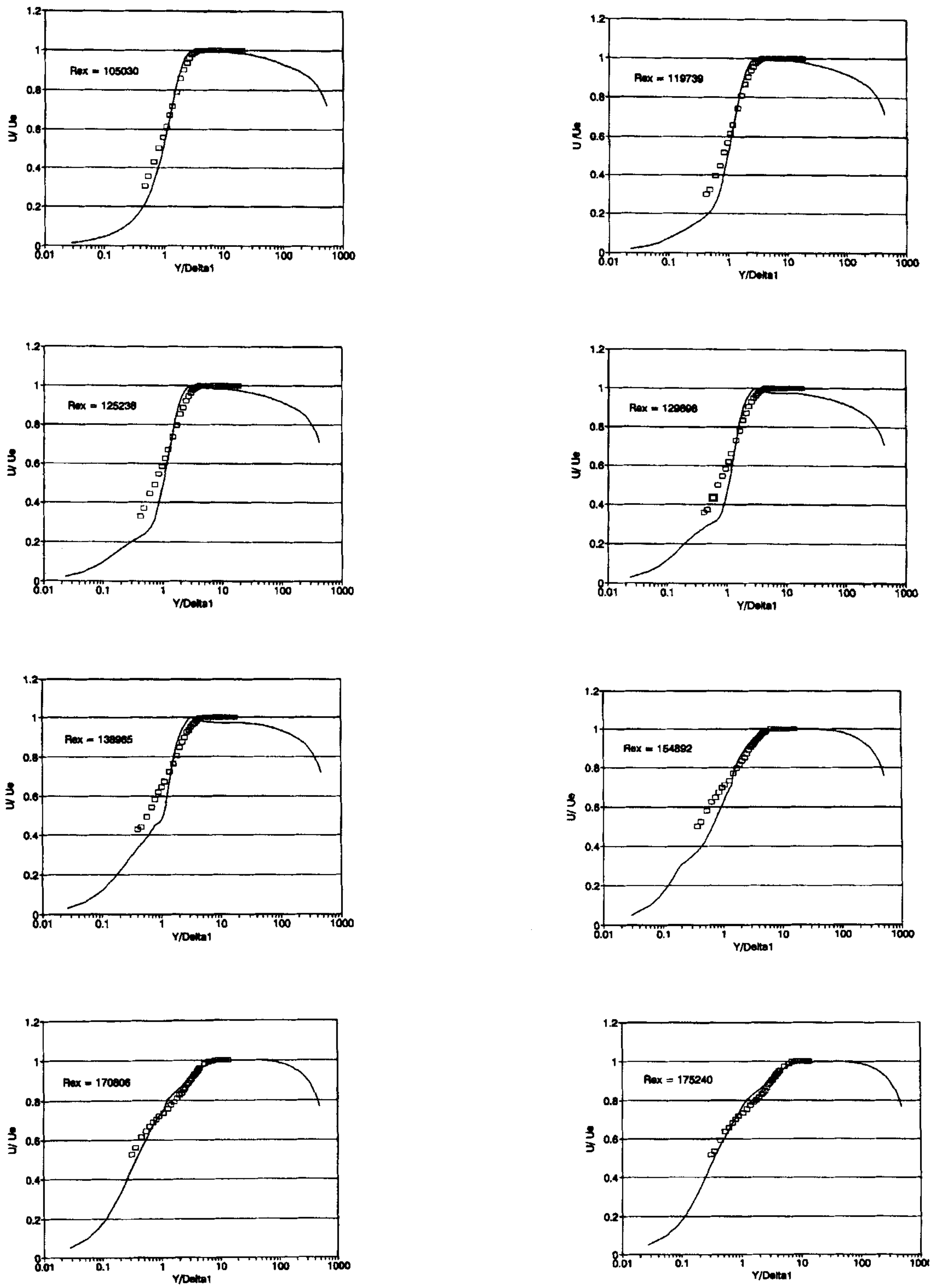

Figure 16. Mean velocity profiles at eight locations along plate for SUG5K6 (□, experimental values)

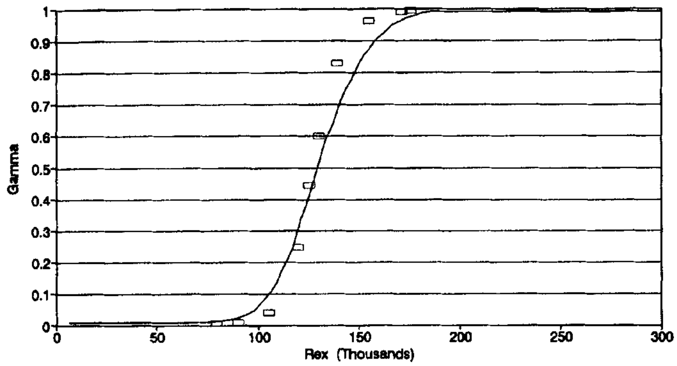

Figure 17. Evolution of intermittency factor γ along plate for SUG5K6 (□, experimental values)

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